Mathematical discussion on a circumstellar matter model for massive stars

Rodrigo Meneses[1], **Dieter H. Nickeler**[2]

[1]Escuela de Ingeniería Civil, Universidad de Valparaíso, General Cruz 222, Valparaíso, Chile
[2]Astronomical Institute AV CR Ondřejov, Fričova 298, 25165 Ondřejov, Czech Republic

rodrigo.meneses@uv.cl, dieter.nickeler@asu.cas.cz

Universidad de Valparaíso CHILE
Astronomical Institute of the Czech Academy of Sciences

Introduction

The correct selection of the governing equations to describe structures in circumstellar matter around massive stars is fundamental for modeling the observed phenomenon. This kind of scenario arises in problems where it's necessary to model an interaction term through a constitutive rule, for example the density. From the fact that the model response must be able to also describe astronomical observations, the theoretical approach should maintain both rigor and adaptability. In this work a discussion on the formulation of mathematical stationary two-dimensional models in a magneto-hydrodynamic framework is presented.

This Work

- We continue the discussion presented in [3] on the use of classical analytical tools to explain the occurrence of circumstellar matter structures.
- The aim corresponds to describing fundamental aspects of the velocity field that can be useful for analyzing structures around B[e] supergiants.
- The focus lies on the impact of the representation used for terms describing the contributions due to pressure and gravity.
- The problem is reduced to the analysis of a Poisson-like differential equation described for a 2-dimensional stationary stream function.
- Starting with a linear analysis to identify potential challenges in the modeling process, we address the nonlinear model formulation by determining the shape in which gravity and pressure terms interact.

Case Study: LHA 120-S 73

- In [1] is indicated that observation of LHA 120-S 73 with the *Spitzer* Space Telescope IRS revealed an intense mid infrared excess emission with clear indication of amorphous silicate dust.
- Ring distances for $M_* = 27M_\odot$

Ring No.	Element(s)	v_{rot} (km s^{-1})	R (AU)
1	[Ca II] ([O I] λ 577)	39	15.7
2	[O I] λ 6300, CO	34	20.7
3	[Ca II]([O I] λ 5577)	22	49.5
4	[O I] λ 6300	16	93.6

- The objective is to fit the observational data presented in the previous table to a mathematical model that allows for the development of descriptions of the gas mass and velocity.

Basic equations

- The governing equations read as follows:

$$\nabla\cdot(\rho\mathbf{v}) = 0, \quad \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p + \rho\mathbf{g} \tag{1}$$

in which ρ is the mass density, $\mathbf{v}$ the gas velocity, p is the gas pressure and $\mathbf{g}$ is the gravitational acceleration of the star with the gravitational potential ϕ. We assume a 2D scenario and $\nabla\cdot\mathbf{v} = 0$.
- Considering $\mathbf{w} = \sqrt{\rho}\mathbf{v}$, the incompressibility assumption implies $\nabla\cdot\mathbf{w} = 0$. Thus, a scalar function ψ can be introducing from the relationship $\mathbf{w}(x,y) = \nabla\psi(x,y)\times\mathbf{e}_z$.
- The model formulation is developed considering the total pressure $\Pi(x,y) = p + \frac{1}{2}\|\mathbf{w}\|^2$. See [3] for further details.
- Under the assumption that $\nabla\psi$ and $\nabla\psi$ are linearly independent, it can be proven that $\Pi_0 = \Pi_0(\psi)$ satisfying

$$\frac{\partial\Pi}{\partial\psi} = \Delta\psi, \quad \frac{\partial\Pi}{\partial\phi} = -\rho(\psi) \tag{2}$$

- Thus, integrating the second relation in (2) and inserting in the first one follows

$$\boxed{\Delta\psi = -\rho'(\psi)\phi + \Pi_0'(\psi)} \tag{3}$$

where the functions $\rho(\psi)$ and $\Pi_0(\psi)$ are not known, and therefore they must be proposed for the development of the analysis.
- Eq. (3) is called the Grad-Shafranov equation and corresponds to the model used in the present work. The type of dependency of the functions with respect to ψ leads to either linear or nonlinear Poisson-type equations.

Preliminaries results: Linear Analysis

- Proposal:

$$\rho(\psi) = \rho_\infty + a_0\psi + \frac{a_1}{2}\psi^2, \quad \Pi_0(\psi) = b_\infty + b_0\psi + \frac{b_1}{2}\psi^2 \tag{4}$$

- The contribution $\Phi \sim R^{-1}$, with $R = \sqrt{x^2+y^2}$, induce the transformation $x = R\cos(\theta)$ and $y = R\sin(\theta)$. The model is expressed as follows

$$R\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial\theta^2} = (GM_*a_0R + b_0R^2) + (GM_*a_1R + b_1R^2)\,\psi \tag{5}$$

with a_i and b_i for $i = 0, 1$ free parameters.

- The solution is constructed considering $\psi(R,\theta) = \psi_h(R,\theta) + \psi_p(R)$ with $\psi_p(R)$ a particular radial solution of (5).
- Similar than [2], the problem is treated considering the ansatz $\psi_h(R,\theta) = f(R)g(\theta)$, from which are deduced (with $\gamma\in\mathbb{C}$)

$$R\frac{d}{dR}\left(R\frac{df}{dR}\right) - \left(GM_*a_1\,R + b_1R^2\right)f = -\gamma\, f, \tag{6a}$$

$$\frac{d^2g}{d\theta^2} = \gamma\, g \tag{6b}$$

- Necessary regularity constraint yield us to impose periodic conditions $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$. In this way, a Sturm-Liouville problem is formulated from which is concluded that $\gamma = -k^2$ with $k = 0, 1, \ldots$. Thus, for each $k\in\mathbb{N}$ we have

$$g_k(\theta) = A_k\cos(k\theta) + B_k\sin(k\theta) \tag{7}$$

- The solution is expressed from the superposition of each contribution as follows

$$\psi_h(R,\theta) = A_0f_0(R) + \sum_{k=1}^{\infty} A_kf_k(R)\cos(k\theta) + B_kf_k(R)\sin(k\theta)$$

where the coefficients A_k and B_k are new free parameters.
- To obtain conditions for the parameters, we consider the gauge transformation $f_k(R) = \hat{r}^{-1/2}Z_k(\hat{r})$, with $\hat{r} = R/R_*$, from which Eq. (6a) becomes

$$\frac{d^2Z}{d\hat{r}^2} - \left(\beta_1 + \frac{\alpha_1}{\hat{r}} - \frac{\frac{1}{4}-k^2}{\hat{r}^2}\right)Z = 0 \tag{8}$$

with $\alpha_1 = GM_*R_*$ and $\beta_1 = b_1R_*^2$.
- Applying comparison results on (8), we conclude that there are oscillatory solution under the assumption $\beta_1 < 0$ or when $\beta_1 = 0$ together with $\alpha_1 < 0$.
- For the case $\beta_1 = 0$ and $\alpha_1 < 0$, taking $\psi_p(R) = a + bR$, directly follows $\psi_p(R) = -\frac{a_0}{a_1} - \frac{b_0}{GM_*a_1}R$.
Thus, the constraint for the existence of bounded solution is given by $b_0 = 0$, which correspond to the case $\Pi_0'(\psi) = 0$.
Moreover, the homogeous part is solved considering by $\eta = \sqrt{-4\alpha_1\hat{r}}$ and $W(\eta) = \psi(\hat{r})$. Employing into the radial ODE in Eq. (6a) we obtain $\eta^2W'' + \eta W' + \left(\eta^2 - (2k)^2\right)W$, and therefore an asymptotic description of the form $f_k(R) \sim \frac{1}{R^{1/4}}G(R)$ with $G(R)$ a superposition of $\sin$ and $\cos$ functions.
- From the previous analysis, the only admissible configuration correspond to $\beta_1 < 0$. This point was also proven through asymptotic analysis and exact solutions, which are related to Coulomb wave functions

Main results: Non-Linear Model

- The expression for the density and the presure contribution are given by

$$\rho(\psi) = \rho_\infty + \frac{\alpha}{\beta}e^{\beta\psi}, \quad \Pi_0(\psi) = \Pi_\infty + \frac{\alpha_2}{2\beta}e^{2\beta\psi} \tag{9}$$

- Considering $\widetilde{\psi}(\zeta,\theta) = \beta\psi + \ln(R)$ the model is reduced as follows

$$\boxed{\frac{\partial^2\widetilde{\psi}}{\partial\zeta^2} + \frac{\partial^2\widetilde{\psi}}{\partial\theta^2} = \tilde{\alpha}_1e^{\widetilde{\psi}} + \tilde{\alpha}_2e^{2\widetilde{\psi}}} \tag{10}$$

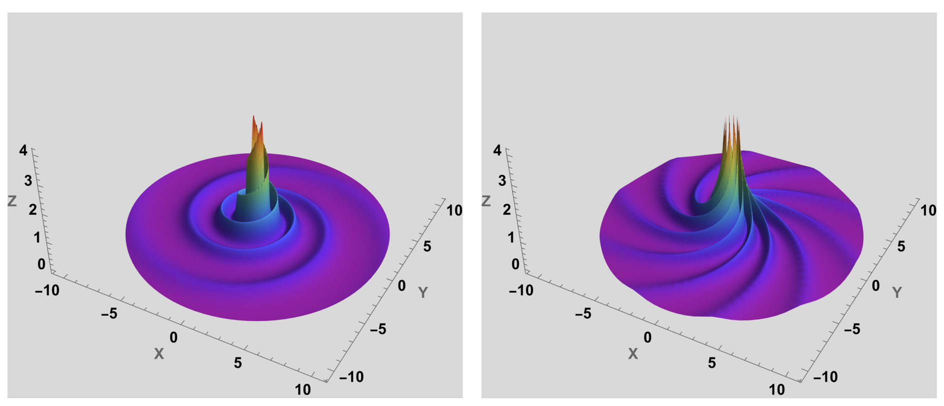
with $\zeta = \ln(R)$, $\tilde{\alpha} = GM_*\alpha\beta$ and $\tilde{\alpha}_2 = \alpha_2\beta$. Writing

$$\tilde{\alpha}_1e^{\tilde{\psi}} = (\tilde{\alpha}_1 + \tilde{\alpha}_1\psi + \ldots), \quad \tilde{\alpha}_2e^{2\tilde{\psi}} = (\tilde{\alpha}_2 + 2\tilde{\alpha}_2\psi + \ldots) \tag{11}$$

from the linear analyisis follows that $\tilde{\alpha}_2 < 0$. Related to the density description, we select $\tilde{\alpha}_1 > 0$ to the analysis.
- Identifying $\tilde{\alpha}_1 > 0$ and $\tilde{\alpha}_2 < 0$, we select the traveling-wave solution presented in [4] Section 9.2.1 (see formula 9.2.1.2.3°), obtaining

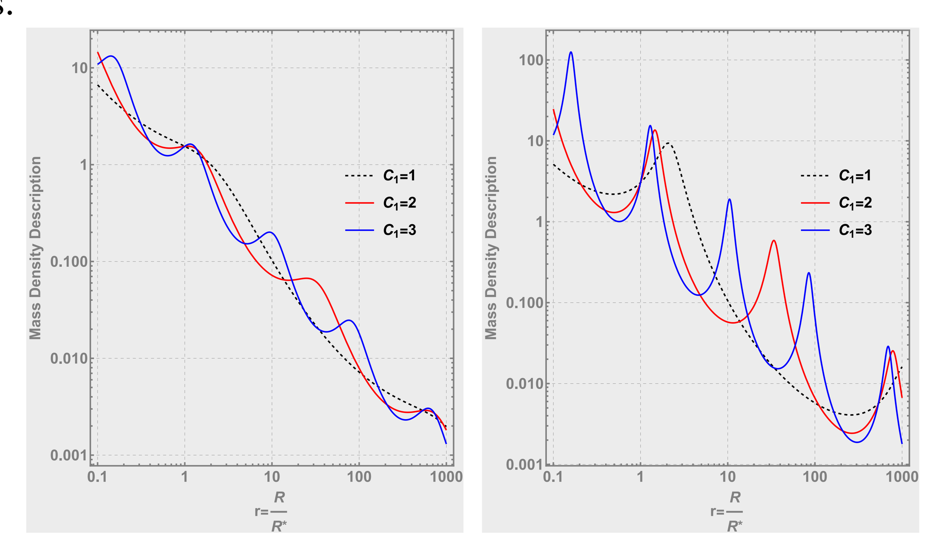
$$e^{\beta\psi(x,y)} = \frac{C_1}{(1+\gamma\,\sin(C_1\ln(R) + C_2\theta + C_3))\,R} \tag{12}$$

with C_i free parameters, and $\gamma = \sqrt{\tilde{\alpha}_1^2 + \tilde{\alpha}_2(C_1^2 + C_2^2)}/\tilde{\alpha}_1$.



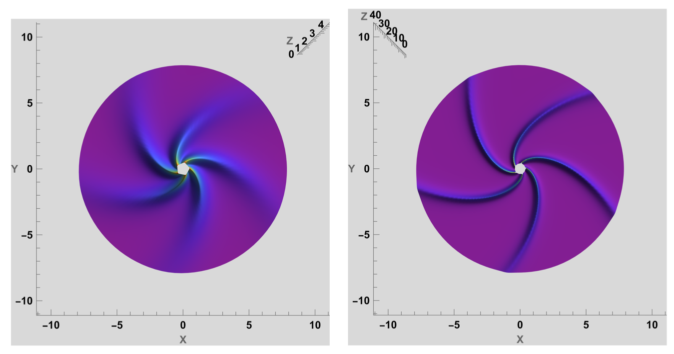
$C_2 = 2$ $\qquad$ $C_2 = 10$

- Various aspects can be described based on the selection of parameters.



γ=0.5 $\qquad$ γ=0.95

Regarding the study related with the parametric dependence, in this figure are presented the Log-Log plots of $e^{\beta\psi}$. It is presented the behavior with respect to the radius and the incidence of γ. Cases γ =0.5 and γ =0.95.



γ =0.5 $\qquad$ γ=0.95

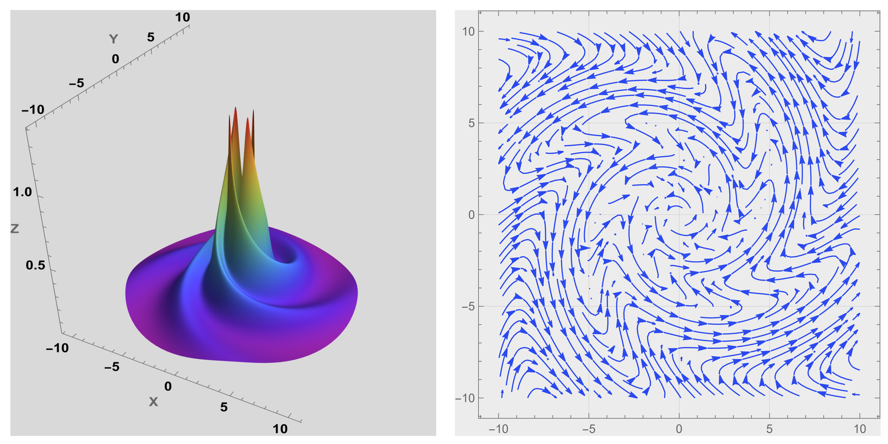
- Considering the quasi-Keplerian expected behavior of the orbits, we select $\rho(\psi) = (\alpha/\beta)e^{\beta\psi}$, obtaining

$$\|\boldsymbol{v}\|^2 = \frac{g(R,\theta)(1 + C_1G(R,\theta)^2 + (C_2G(R,\theta))^2)}{\beta(C_1^2 + C_2^2)\,R} \tag{13}$$

for $r(R,\theta)$ and $G(R,\theta)$ oscillatory functions, satisfying $\|\boldsymbol{v}\|^2 \sim R^{-1}$ for R large.

Application on the LHA 120-S 73 data

- The procedure is based on the following steps:
Step 1 Find C_1, C_2 and γ from $\frac{\partial\rho}{\partial R} = 0$.
Step 2 For these values, fit β using (13).



Density-shape $\qquad$ Gas-velocity

- Description obtained for the values $(C_1, C_2, \gamma, \beta)$ =(7.20682,-5,0.2529,6.68758$\times10^{-6}$)

Conclusions and Future Directions

- The linear approach allows identifying essential relationships in the model parameters, and therefore distinguishing constraints in more complex models.
- The parametric analysis of the solution enhances the understanding of the problem.
- From the solutions of the linear and non-linear versions of the Grad-Shafranov equation, the expected behaviors of the physical problem are described.
- It was possible to fit the parameters of the mathematical solution to the observational data. The procedure can be applied to the process of describing other structures in circumstellar matter.
- It is necessary to analyze the procedure for fitting the mathematical model more rigorously given the limited observational information.
- The treatment of temporal dependency is one of the works that we will address soon.

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