Local Dynamics at the Apex of an Astropause: Insights from 3D Resistive MHD Solutions

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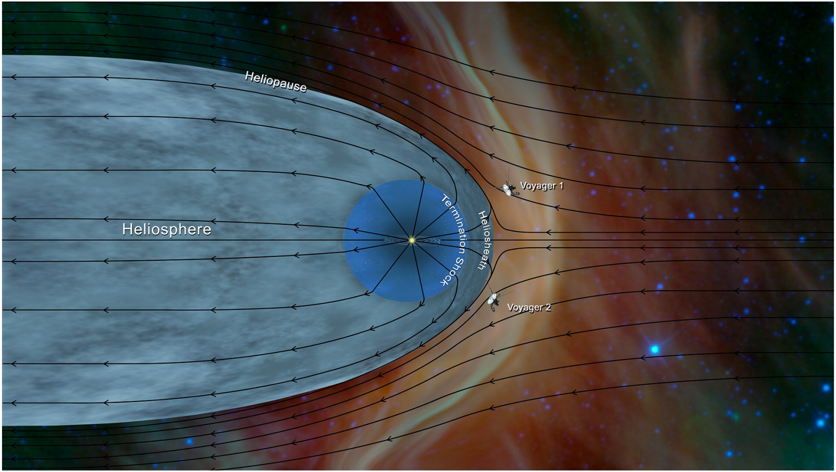
Motivation



Credits: NASA JST

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Magnetohydrodynamics

- Describes the dynamics of a conducting fluid in the presence of electromagnetic fields.

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\
&\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} \\
&\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \\
&\nabla \cdot \mathbf{B} = 0
\end{aligned}
$$

- It is useful to express the equations in terms of dimensionless quantitites. Independent variables and differential operators:

$$\bar{l} = l/l_0, \quad \bar{t} = t/t_0 \qquad \Rightarrow \qquad \bar{\nabla} = l_0 \nabla, \quad \partial/\partial \bar{t} = t_0 \partial/\partial t$$

- Dimensionless dependent variables:

$$\bar{\rho} = \rho/\rho_0 \quad \bar{\mathbf{v}} = \mathbf{v}/v_0 \quad \bar{p} = p/(\rho_0 v^2) \quad \bar{\mathbf{B}} = \mathbf{B}/B_0$$

Linear Flows

- Fields close to the null points can be Taylor expanded.
- Assumption: The velocity null and the magnetic null coincides.

- Taylor Series for a scalar:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- Taylor Expanding fields:

$$\mathbf{v} = \mathcal{V} \cdot \mathbf{r}$$

$$\mathbf{B} = \mathcal{B} \cdot \mathbf{r}$$

$$p = \mathbf{r}^T \cdot \mathcal{P} \cdot \mathbf{r} + p_0$$

where

$$\mathbf{r} = \begin{pmatrix} x & y & z \end{pmatrix}^T$$

$$\mathcal{V}_{ij} = \frac{\partial v_i}{\partial x_j} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}$$

$$\mathcal{B}_{ij} = \begin{pmatrix} 1 & \frac{(q-j_{\parallel})}{2} & 0 \\ \frac{(q+j_{\parallel})}{2} & b & 0 \\ 0 & j_{\perp} & -(b+1) \end{pmatrix}$$

$$\mathcal{P}_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j}$$

3D Linear Dynamical System

- Equations of the form $\mathbf{v} = \mathcal{V} \cdot \mathbf{r}$, where $\mathcal{V}$ has three eigenvalues λ_1, λ_2 and λ_3.
- Saddle Points
  - Not all eigenvalues have real parts with the same sign.
  - Topological skeleton has two components: Fan plane and the Spine line.
  - Fields moving towards the null point: Stable subspace, and fields moving away from the null: Unstable subspace.
  - The null is radial if all eigenvalues are real, and is a spiral if two of the eigenvalues are complex conjugates.

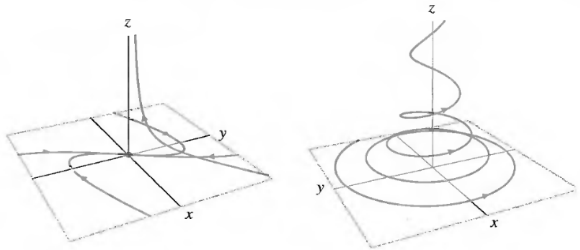


Figure: (a) Radial null and (b) Spiral null

3D Magnetic Field Structure

- The magnetic field is given by $\mathbf{B} = \mathcal{B} \cdot \mathbf{r}$, where the matrix $\mathcal{B}$ is prescribed as:

$$\mathcal{B} = \begin{pmatrix} 1 & \frac{(q-j_{\parallel})}{2} & 0 \\ \frac{(q+j_{\parallel})}{2} & b & 0 \\ 0 & j_{\perp} & -(b+1) \end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix} j_{\perp} \\ 0 \\ j_{\parallel} \end{pmatrix}$$

- To determine the topological structure, we need to compute the eigenvalues.

$$\lambda_{B_{1,2}} = \frac{1}{2}(b+1) \pm \frac{1}{2}\sqrt{j_c^2 - j_{\parallel}^2} \quad \lambda_{B_3} = -(b+1) \qquad \begin{aligned} b &\geq -1 \\ (b+1)^2 &\geq j_c^2 - j_{\parallel}^2 \\ j_c^2 &= (b-1)^2 + q^{2`} \end{aligned}$$

- Note that as $\mathcal{B}$ is traceless, the magnetic field is always a saddle point.

Titov and Hornig (2000)

- Solved the stationary MHD equations locally for constant resistivity.
- Flow is considered to be incompressible ($\nabla \cdot \mathbf{v} = 0$)

- Stationary MHD equations:

$$\mathcal{V}^2 - \mathcal{V}^{T2} = \mathcal{B}^2 - \mathcal{B}^{T2} \qquad (1)$$

$$\mathrm{tr}(\mathcal{V}) = 0 \qquad (2)$$

$$\mathcal{V}\mathcal{B} - \mathcal{B}\mathcal{V} = 0 \qquad (3)$$

$$\mathrm{tr}(\mathcal{B}) = 0 \qquad (4)$$

- The pressure is given by:

$$\mathcal{P} = \mathcal{B}^2 - \mathcal{B}^T\mathcal{B} - \mathcal{V}^2 \qquad (5)$$

- (2) and (3) $\Rightarrow$

$$\mathcal{V}_{12} = (j_{\parallel} - q)(\mathcal{V}_{11} - \mathcal{V}_{22})/[2\,(b-1)], \qquad (A1)$$

$$\mathcal{V}_{13} = 0, \qquad (A2)$$

$$\mathcal{V}_{21} = (j_{\parallel} + q)(\mathcal{V}_{22} - \mathcal{V}_{11})/[2\,(b-1)], \qquad (A3)$$

$$\mathcal{V}_{23} = 0, \qquad (A4)$$

$$\mathcal{V}_{31} = \frac{6 j_{\perp}(j_{\parallel} + q)(\mathcal{V}_{22} - b\mathcal{V}_{11})}{(b-1)(j_{\parallel}^2 + 8b^2 + 20\,b + 8 - q^2)}, \qquad (A5)$$

$$\mathcal{V}_{32} = j_{\perp}\{[4\,(b^2 + b - 2) - j_{\parallel}^2 + q^2]\mathcal{V}_{11} + [8\,(b^2 + b - 2) + j_{\parallel}^2 - q^2]\mathcal{V}_{22}\}/(b-1)(j_{\parallel}^2 + 8b^2 + 20b + 8 - q^2), \qquad (A6)$$

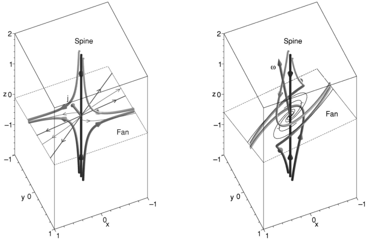
$$\mathcal{V}_{33} = -(\mathcal{V}_{11} + \mathcal{V}_{22}). \qquad (A7)$$

Titov and Hornig (2000)

- The momentum equation (1) is a skew-symmetric matrix, i.e. three equations for 2 parameters.

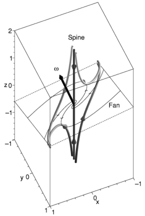
  - Field-Aligned (FA) flows:

$$V_{11} = \pm 1 \quad V_{22} = \pm b \quad \Rightarrow \mathcal{V} = \pm \mathcal{B}$$



  - Spiral field-crossing (SFC) flows:

$$V_{11} = \pm(1 - b^2 + S^2)/(2S) \quad S^2 = (j_{\|}^2 - j_c^2)/3$$
$$V_{22} = \pm(b^2 - 1 + S^2)/(2S)$$



- The fan and spine for both fields always coincide.

MHD Equations Revisited

Full MHD equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B}$$
$$\nabla \cdot \mathbf{v} = 0$$
$$\nabla \times \mathbf{B} = \mathbf{j}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

Stationary case for the Linearized fields:

$$\mathcal{V}^2 - \mathcal{V}^{T2} = \mathcal{B}^2 - \mathcal{B}^{T2} \quad (6)$$
$$\mathrm{tr}\,(\mathcal{V}) = 0 \quad (7)$$
$$\nabla \times \mathbf{E} = 0 \quad (8)$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \quad (9)$$
$$\mathrm{tr}\,(\mathcal{B}) = 0 \quad (10)$$

The pressure can be calculated as:

$$\mathcal{P} = \mathcal{B}^2 - \mathcal{B}^T\mathcal{B} - \mathcal{V}^2 \quad (11)$$

Methods

- The Electric field can be written as a gradient of a scalar function.

$$\nabla \times \mathbf{E} = 0 \quad \Longrightarrow \quad \mathbf{E} = -\nabla\phi \tag{12}$$

- The Ohm's law is a set of three equations, one for each component.

$$-\frac{\partial\phi(x,y,z)}{\partial x} + v_y B_z - v_z B_y = \eta(x,y,z) j_\perp \tag{13a}$$

$$-\frac{\partial\phi(x,y,z)}{\partial y} + v_z B_x - v_x B_z = 0 \tag{13b}$$

$$-\frac{\partial\phi(x,y,z)}{\partial z} + v_x B_y - v_y B_x = \eta(x,y,z) j_\| \tag{13c}$$

- The uniqueness of $\eta(x,y,z)$, the condition of incompressibility, and the momentum equation provides us with 7 constraints for 9 velocity matrix elements.

Solutions

- We choose the independent parameters to be V_{11} and V_{13}.
- The velocity matrix can now be expressed in terms of the magnetic field parameters and two free paranemets, namely V_{11} and V_{13}.

$$\mathcal{V} = \mathcal{V}(V_{11}, V_{13}, b, q, j_{\|}, j_{\perp}) \tag{14}$$

- The solutions can be categorized into four classes.

	V_{11}	V_{13}
class I	0	0
class II	1	0
class III	0	1
class IV	1	1

Class II: $V_{13} = 0$

$$\mathcal{V}_1(V_{11}, b, q, j_{\|}, j_{\perp}) = \begin{pmatrix} V_{11} & \frac{(q-j_{\|})}{2} & 0 \\ \frac{(q+j_{\|})}{2} & \frac{3j_{\|}(1+b-V_{11})+q(-1+b+V_{11})}{(q+3j_{\|})} & 0 \\ 0 & j_{\perp} & \frac{-3(1+b)j_{\|}-q(-1+b+2V_{11})}{(q+3j_{\|})} \end{pmatrix} \quad (15)$$

$$\mathcal{V}_2(V_{11}, b, q, j_{\|}, j_{\perp}) = \begin{pmatrix} V_{11} & -\frac{(q-j_{\|})}{2} & 0 \\ -\frac{(q+j_{\|})}{2} & \frac{-3j_{\|}(1+b+V_{11})+q(1-b+V_{11})}{(q+3j_{\|})} & 0 \\ 0 & -j_{\perp} & \frac{3(1+b)j_{\|}+q(-1+b-2V_{11})}{(q+3j_{\|})} \end{pmatrix} \quad (16)$$

- They have a similar form as the magnetic field.

$$\mathcal{B} = \begin{pmatrix} 1 & \frac{(q-j_{\|})}{2} & 0 \\ \frac{(q+j_{\|})}{2} & b & 0 \\ 0 & j_{\perp} & -(b+1) \end{pmatrix} \quad (17)$$

Field-Aligned Flows

- We consider the special case where $V_{11} = 1$ or $V_{11} = -1$.

$$\mathcal{V}_1(1,0) = -\mathcal{V}_2(-1,0) = \mathcal{B} \tag{18}$$

$$\mathcal{V}_1(-1,0) = -\mathcal{V}_2(1,0) = \begin{pmatrix} -1 & \frac{(q-j_{\|})}{2} & 0 \\ \frac{(q+j_{\|})}{2} & -2+b+\frac{12j_{\|}}{q+3j_{\|}} & 0 \\ 0 & j_{\perp} & 3-b-\frac{12j_{\|}}{q+3j_{\|}} \end{pmatrix} \tag{19}$$

- If we fix $c_1 = c_2 = 0$ for our ansatz, we get constant resistivity.

$$\eta(x,y,z) = c_3$$

Example: Radial Null

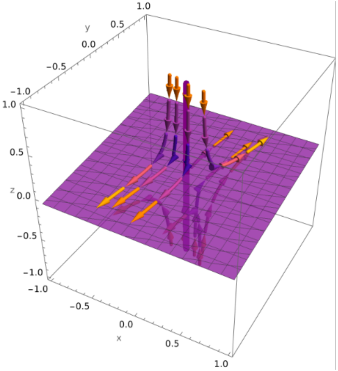
- We investigate the dependence of solutions $\mathcal{V}_1$ on the free parameter V_{11} for specific magnetic field configurations.
- For the first example, we fix the magnetic field parameters to be $b = 5, q = 3$, $j_\perp = 1$ and $j_\parallel = 1$. This configuration corresponds to a radial magnetic null.

$$\mathcal{B} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 0 \\ 0 & 1 & -6 \end{pmatrix}$$

$$\lambda_{B_1} = 5.44949$$

$$\lambda_{B_2} = 0.55051$$

$$\lambda_{B_3} = -6.0$$



Example: Radial Null

- the Velocity field takes the simplified form

$$\mathcal{V}_1(V_{11}) = \begin{pmatrix} V_{11} & 1 & 0 \\ 2 & 5 & 0 \\ 0 & 1 & -5 - V_{11} \end{pmatrix}$$

- The eigenvalues and eigenvectors are given as:

$$\lambda_{V1} = 0.5(5 + V_{11} - \sqrt{33 - 10V_{11} + V_{11}^2}) \qquad \lambda_{V2} = 0.5(5 + V_{11} + \sqrt{33 - 10V_{11} + V_{11}^2}) \qquad \lambda_{V3} = -5 - V_{11}$$

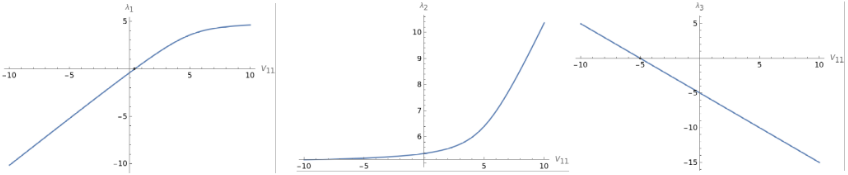


Figure: The parametic dependence of each eigenvalue on V_{11}

- The velocity field is always a radial null.

Example: Radial Null

- Case 1: $V_{11} < -5$.
  - $\lambda_{V_1} < 0$, $\lambda_{V_2} > 0$, $\lambda_{V_3} > 0$.
  - Spine lies along $\xi_{\mathbf{1}}$: Stable subspace.

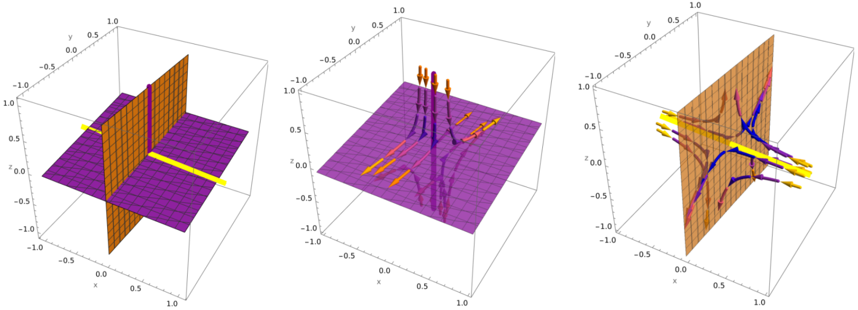


Figure: (a) Orientation of Separatrices, (b) Structure of ***B*** and (c) Structure of ***v***

- Case 2: $V_{11} = -5$. Degenerate case.

Example: Radial Null

- Case 3: $-5 < V_{11} < 0.4$.
  - $\lambda_{V_1} < 0$, $\lambda_{V_2} > 0$, $\lambda_{V_3} < 0$.
  - Spine lies along ξ_2 : Unstable subspace.

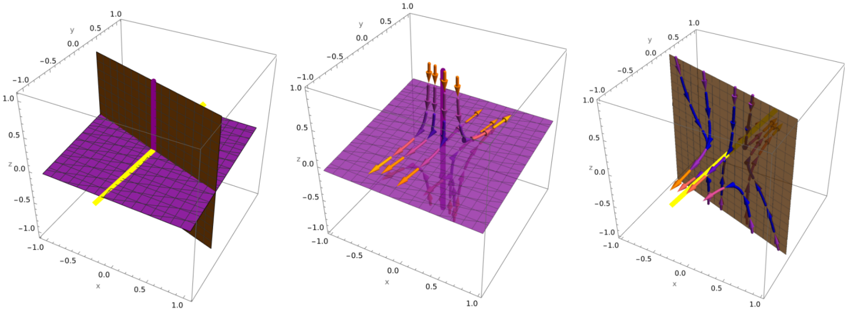


Figure: (a) Orientation of Separatrices, (b) Structure of $\boldsymbol{B}$ and (c) Structure of $\boldsymbol{v}$

- Case 4: $V_{11} = 0.4$. Degenerate case.

Example: Radial Null

- Case 5: $V_{11} > 0.4$. Spine lies along ξ_3 : Stable subspace.
  - $\lambda_{V_1} > 0, \lambda_{V_2} > 0, \lambda_{V_3} < 0$
  - Spine lies along ξ_3 : Stable subspace.

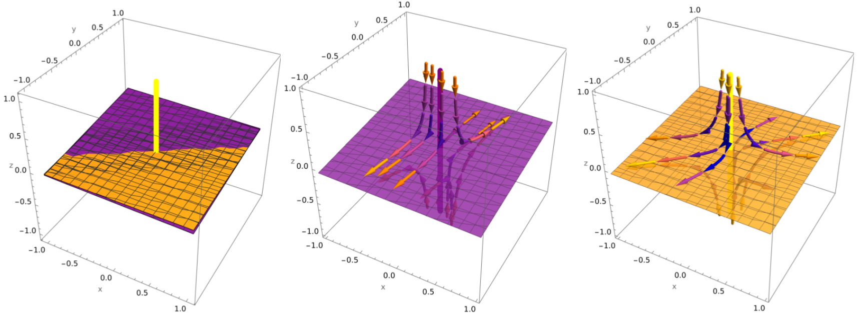


Figure: (a) Orientation of Separatrices, (b) Structure of ***B*** and (c) Structure of ***v***

Spiral structure for **v**?

- The characteristic polynomial of $\mathcal{V}_1(V_{11}, b, q, j_{||}, j_{\perp})$ can be written in the following form:

$$\left(x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2\right)(x - \lambda_3) = 0$$

- For the roots to be complex, the discriminant must be negative.

$$\frac{3j_{||}(1+b) + q(-1+b) - S}{6j_{||}} < V_{11} < \frac{3j_{||}(1+b) + q(-1+b) + S}{6j_{||}}$$

where $S = (3j_{||} + q)\sqrt{j_{||}{}^2 - q^2}$

- The condition for existance of spiral nulls

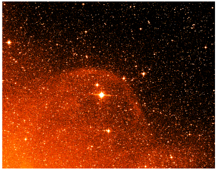
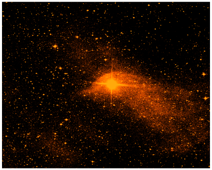
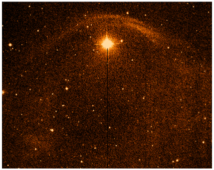
$$j_{||} > |q| \quad \text{or} \quad j_{||} < -|q|$$

Future Work

- The estimate of pressure, along with the characteristic values of density, can give us the emission measure in say Hα (e.g. Rybicki and Lightman, 1979).

$$EM \propto \text{coeff. of transition}(T_e) \times n_e^2 \approx T_e n_e^2$$

- 3D MHD simulations to compute synthetheic skymaps and estimate the emission.
- Comparison with observations. (Our Hα survey on BSG bow shock structures)

Conclusions

- The linearized equations of resistive MHD were solved analytically in a local region close to the null point where both velocity field and magnetic field are zero.
- The solutions can be described in terms of the magnetic field parameters, and two free parameters.
- As the next step, we would like to find a better method of categorize the solutions.
- Emission measure can be calculated by estimating the pressure, density, and hence temperature. This can be then compared to the results from simulations and observations.